Optimum Values and Response Surfaces

1. Physical Optimum Values

Suppose in the production of a crop, say wheat, the input factors are the levels or doses of *N*, *P* and *K* expressed in kg/ha and the output factor is yield of the crop (y) expressed in Quintals/ha. It is known that value of y shall be different for different combinations of the values of input factors. Out of these different combinations of values of input factors, there will be a combination for which the value of output factor y shall be maximum. Let us suppose that p kg/ha of *N*, q kg/ha of *P* and r kg/ha of *K* applied together gives us m quintals per hectare of yield (y) of the crop which is the maximum of all the values of y obtained from all possible combinations of the values of *N*, *P* and *K*. It means that no other combination of the values of *N*, *P* and *K* gives yield more than m quintals/ha. Then the values p, q and r are called the **physical optimum of the input factors** *N*, *P* and *K* respectively and the value m is called **physical optimum of** y for these values of *N*, *P* and *K*.

2. Economic Optimum Values

If the cost price of the input factors and selling price of the output factor are considered along with their quantitative values, then different quantitative combinations of input factors shall give different values of the profit (= selling price of y − cost price of *N*, *P* and *K*) that may be obtained in the production of y. Suppose x, y and z (kg/ha) are the quantitative values or doses of *N*, *P* and *K* to obtain the maximum profit (Rs. *B*/ha), then x, y and z are called the **economic optimum** doses for the input factors *N*, *P* and *K* respectively.

3. Response Surface

A **response surface** is the function f representing the relationship of output or response factor y with input factors $x_1, x_2, \ldots, x_n$. Thus if

$$y = f(x_1, x_2, \ldots x_n) + e \qquad \ldots(1)$$

then f is the response surface and e denotes the experimental error. The experimental error arises because of the difference in the observed value of y (from an experiment) and its calculated value (from eq. 1).

Generally the actual functional relationship expressed by eq. 1 is hardly known and is, therefore, approximated by a properly chosen function, such as polynomial, exponential etc.

In order to determine the response surface or functional relationship given by eq. 1. We fit the given data for $y, x_1, x_2, \ldots, x_n$ to a propely chossen function by any one of the methods like method of least squares or method of orthogonal polynomials. For example we may fit the first or second degree polynomial to one input factor x_1 by assuming the function as

$$y = a_0 + a_1x_1 \qquad \text{(for first degree polynomial)}$$

$$y = a_0 + a_1x_1 + a_2x_1^2 \qquad \text{(for second degree polynomial)}$$

Similarly we may assume the following function for two input factors x_1 and x_2

$$y = a_0 + a_1x_1 + a_2x_2 \qquad \text{(for first degree polynomial)}$$

$$y = a_0 + a_1x_1 + a_2x_2 + a_2x_1^2 + a_4x_2^2 + a_5x_1x_2 \qquad \text{(for second degree polynomial)}$$

While fitting any function to the data we may use the data as given (in original units) or transform it by using any of the coding procedure for the sake of convenience.

4. Method of Determination of Physical Optimum

The following procedure is adopted for this purpose.

Step-1. We fit the given data for response or output factor y and one input factor x or n input factor $x_1, x_2, ..., x_n$ properly choose function as

$$y = f(x), \quad \text{(for one input factor } x) \qquad ...(1)$$

or

$$y = f(x_1, x_2, ..., x_n) \quad \text{(for } n \text{ input factors } x_1, x_2, ... x_n) \qquad ....(2)$$

Step-2. Now we differentiate y with respect to x for one input factor x and with respect to $x_1, x_2, ..., x_n$ for n input factors and equate to zero each one of these values of differential coefficients to get the following equation :

One equation–

$$\frac{dy}{dx} = \frac{df(x)}{dx} = 0, \qquad \text{for one input factor}$$

or

***n* equations–**

$$\left.\begin{aligned} \frac{dy}{dx_1} &= \frac{df(x_1, x_2, ... x_n)}{dx_1} = 0 \\ \frac{dy}{dx_2} &= \frac{df(x_1, x_2, ... x_n)}{dx_2} = 0 \\ &\vdots \\ \frac{dy}{dx_n} &= \frac{df(x_1, x_2, ... x_n)}{dx_n} = 0 \end{aligned}\right\} \quad \text{for } n \text{ input factors}$$

Step-3. **(a)** Solve one equation obtained in step-2 for one input factor to get the value of x which is physical optimum of input x.

or **(b)** Solve n equation obtained in step – 2 for n input factors to get values of $x_1, x_2, ... x_n$. that are physical optimum of the corresponding input factors.

Step-4. **(a)** For one input factor, substitute the value of x obtained in step 3 (a) in equation 1 obtained in step 1 to get the physical optimum value of output y corresponding to these physical optimum values of input factors.

or **(b)** For n input factors, substitute the values of $x_1, x_2, ... x_n$ obtained in step-3 (b) in eq. 2 to get the physical optimum value of output or response factor y corresponding to these physical optimum values of input factor.

5. Method to Determine–Economic Optimum

The following procedure is followed for this purpose–

Step-1. The given data for out out-put or response factor y and one input factor x or n input factors $x_1, x_2, ..., x_n$ is fitted to the properly chosen function f as

$$y = f(x), \quad \text{(for one input factor } x) \qquad \ldots(1)$$

or

$$y = f(x_1, x_2, \ldots\ldots, x_n.) \qquad \text{(for } n \text{ input factors } x_1, x_2, \ldots\ldots x_n) \qquad \ldots(2)$$

Step-2. (a) Then we differentiate y with respect to x (for one input factor) in eq. (1) and equate it to $\frac{q}{p}$ where q is cost per unit of the input factor x and p is the price per unit of the out-put to get the equation

$$\frac{dy}{dx} = \frac{df(x)}{dx} = \frac{q}{p} \qquad \ldots(3)$$

or **(b)** We differentiate y with respect to $x_1, x_2, \ldots, x_n$ (for n input factors) in eq. (2) and equate the n equations so obtained to $\frac{q_1}{p}, \frac{q_2}{p}, \ldots\ldots \frac{q_n}{p}$ respectively, to get the equations

$$\left.\begin{array}{l} \dfrac{dy}{dx_1} = \dfrac{df(x_1, x_2, \ldots. x_n)}{dx_1} = \dfrac{q_1}{p} \\ \dfrac{dy}{dx_2} = \dfrac{df(x_1, x_2, \ldots. x_n)}{dx_2} = q_2 p \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \dfrac{dy}{dx_n} = \dfrac{df(x_1, x_2, \ldots. x_n)}{dx_n} = \dfrac{q_n}{p} \end{array}\right\} \text{ for } n \text{ input factors} \qquad \ldots(4)$$

where $q_1, q_2, \ldots, q_n$ are the cost per unit of the input $x_1, x_2, \ldots, x_n$ respectively and p is the price per unit of the output or response factor y.

Step-3. (a) Solve one equation obtained in step 2 (a) to get value of x which is the **economic optimum** of input factor x.

or (b) Solve n equations obtained in step 2 (b) to get values of $x_1, x_2, \ldots, x_n$. that are the **economic optimum** of input factors $x_1, x_2, \ldots, x_n$.

Step-4. (a) Substitute value of x obtained in step 3 (a) in eq. (1) obtained in step-1 to get the economic optimum of output y corresponding to this value of input factors.

(b) Substitute values of $x_1, x_2, \ldots x_n$ obtained in step-3 (b) in eq. (2) obtained in step-1 to get the economic optimum of output or response factor y corresponding to these values of input factors.

6. Optimum Values for Quadratic Polynomial

We explain the method when response curve is a quadratic polynomial.

(a) If the equation of response curve is

$$y = a_0 + a_1 x + a_2 x^2,$$

then

$$\frac{dy}{dx} = a_1 + 2a_2 x = 0$$

or

$$x = \frac{-a_1}{2a_2}$$

It is that value of x for which response of y is maximum i.e., it is the value of physical optimum of x.

If the price per unit of out-put or response factor y is p and cost per unit of input factor is q, then for economic optimum value of x, we have

$$\frac{dy}{dx} = a_1 + 2a_2x = \frac{q}{p}$$

or

$$x = \frac{1}{2a_2}\left(\frac{q}{p} - a_1\right)$$

and the maximum net profit $= p(a_1x + a_2x^2) - qx$.

(b) If the equation of response surface for two input factors x_1 and x_2 is

$$y = a_0 + a_1x_1 + a_2x_2 + x_3x_1^2 + a_4x_2^2 + a_5x_1x_2$$

then (i) for physical optimum values of x_1 and x_2 we have

$$\frac{\delta y}{\delta x_1} = a_1 + 2a_3x_1 + a_5x_2 = 0$$

$$\frac{\delta y}{\delta x_2} = a_2 + 2a_4x_2 + a_5x_2 = 0$$

Solving these two equations we get values of physical optimum of x_1 and x_2 and (ii) for economic optimum values of x_1 and x_2, we have

$$\frac{\delta y}{\delta x_1} = a_1 + 2a_3x_1 + a_5x_2 = \frac{q_1}{p}$$

$$\frac{\delta y}{\delta x_2} = a_2 + 2a_4x_2 + a_5x_2 = \frac{q_2}{p}$$

where q_1 and q_2 are cost prices per unit of x_1 and x_2 respectively.

Note : Other types of response curves can also be used in place of polynomials.

Illustration : The values of input factor x and the output factor y are given below :

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	9

Find (a) physical optimum for input factor x.

and (b) the economic optimum for x if the cost of x is Rs. 6 per unit and price of y is Rs. 3 per unit.

Soln. Using the method of least squares explained in the chapter of Curve Fitting of this book, the second degree equation $y = a_0 + a_1x + x_2x^2$ fitted to the give data comes out to be

$$y = -0.919 + 3.41x - 0.256x^2$$

This is the response surface.

(a) For physical optimum of input factor x, we have

$$\frac{dy}{dx} = 3.41 - 0.512\,x = 0$$

or

$$x = \frac{3.41}{0.512} = 6.66 \text{ units}$$

(b) For economic optimum of input factor x, we have

$$\frac{dy}{dx} = 3.41 - 0.512x = \frac{6}{3} = 2$$

or

$$x = \frac{2 - 3.41}{0.512} = \frac{1.41}{0.512} = 2.75 \text{ units}$$

Utility of Response Surfaces

1. We can estimate the response of output factor corresponding to given level(s) of the input factor(s) with the help of response surface.

2. It helps in determining the optimum level of a single factor or optimum levels of a combination of factors for which the response of the output factor is maximum (as in case of yield) or minimum (as in case of cost of production per unit of output).